Propositional Calculus

Propositional Calculus, also called *Sentential Calculus* or *Zeroth Order Logic*, examines the structure of certain fragment common both to natural languages as well as to languages of scientific theories, namely the abstract patterns or forms of propositions or statements occurring in them as well as the relations between them which can be inferred from their form. This will make possible a complete description of valid arguments in terms of structural relations between the forms of their constituents.

We will take advantage of the fact that our readers have already some acquaintance with Propositional Calculus, including the logical connectives and their meaning, truth tables, tautologies, etc. This allows us to focus on some general "philosophical" features of Propositional Calculus which are not always part of the usual courses and use this familiar and rather elementary platform for the exposition of some topics and issues which will reappear later on within the Predicate Calculus in a more advanced form.

Propositions and Propositional Forms

A *propostion* or a *statement* is an affirmative grammatical sentence making a meaningful announcement which is either true or false, no matter whether we or whoever are able to decide its verity. We say that the *truth value* of a proposition is 1 if it is true and it is 0 if it is false.

Given some propositions we can form new ones combining them by means of the unary logical connective *not* and the binary logical connectives *and*, *or*, *if ... then*, *if and only if*, *either ... or*, *neither ... nor*, etc. Propositional Calculus is based on the following fundamental observation:

> *If A is a proposition formed of some simpler propositions $p_1, \ldots, p_n$ by means of logical connectives in a certain way then the truth value of A can be determined just out of the truth values of the propositions $p_1, \ldots, p_n$ and the way how A is formed, regardless of the meaning and content of the propositions $p_1, \ldots, p_n$.*

In other words, the truth value of A can be *computed* from the truth values of its components $p_1, \ldots, p_n$ and the abstract pattern or the *form* of A.

As a consequence, the subject of Propositional Calculus is not primarily propositions themselves but the forms propositions can take on, according to the way how they are composed from simpler propositions by means of logical connectives. These abstract forms we call *propositional forms*; they are expressions (words) of some formal language to be introduced below. In order to describe the *syntax* of the language of Propositional Calculus we will codify its symbols and describe the way how its words are generated.

The *language of Propositional Calculus* has the following symbols divided into three groups:

- Propositional variables: p, q, r, p_0, p_1, p_2, ... , q', q'', ...
- Logical connectives: $\neg$ (*not*), $\wedge$ (*and*), $\vee$ (*or*), $\Rightarrow$ (*if ... then* or *implies*), $\Leftrightarrow$ (*if and only if*) (two would suffice)
- Auxiliary symbols: (,) (*parentheses*) (they could be avoided)

We denote by P the set of all propositional variables. We assume that the set P is countably infinite at least in the potential sense, i.e., whenever we have any finite list of propositional variables $p_1, \dots, p_n$, we are able to find some new propositional variable q not included in that list and, at the same time, all the propositional variables $p \in P$ can be set into a one-to-one correspondence with the natural numbers $n \in \mathbb{N}$.

Propositional forms are certain finite strings, i.e., words, consisting of the above quoted symbols. The set $\mathrm{VF}(P)$ of all propositional forms over the set of propositional variables P is defined recursively as the smallest set containing all the propositional variables and closed with respect to the application of logical connectives, i.e., the smallest set satisfying the following two conditions:

1° $P \subseteq \mathrm{VF}(P)$
(every propositional variable $p \in P$ is a propositional form over the set P)

2° if $A, B \in \mathrm{VF}(P)$ then $\neg A, (A \wedge B), (A \vee B), (A \Rightarrow B), (A \Leftrightarrow B) \in \mathrm{VF}(P)$
(if the strings A, B are propositional forms over the set P then so are the strings $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \Rightarrow B)$ and $(A \Leftrightarrow B)$)

According to 1°, propositional variables are sometimes referred to as *atomic propositional forms*. As a consequence of the fact that the set P of all propositional variables is countable, the set $\mathrm{VF}(P)$ of all propositional forms over P is countable, as well.

The set $\mathrm{VF}(Q)$ of all propositional forms over any nonempty set of propositional variables $Q \subseteq P$ can be defined in an analogous way. In particular, for a finite set $Q = \{p_1, \dots, p_n\}$, we denote

$$\mathrm{VF}(Q) = \mathrm{VF}(p_1, \dots, p_n)$$

Since every propositional form $A \in \mathrm{VF}(P)$ is composed from atomic propositional forms by applying the rule 2° just finitely many times, there always is a finite number of propositional variables $p_1, \dots, p_n \in P$ such that $A \in \mathrm{VF}(p_1, \dots, p_n)$.

If A, B are propositional forms then the propositional form $\neg A$ is called the *negation* of A, and the propositional forms $(A \wedge B)$, $(A \vee B)$, $(A \Rightarrow B)$ and $(A \Leftrightarrow B)$ are called the *conjunction*, the *disjunction* or the *alternative*, the *implication* and the *equivalence* of A and B, respectively.

Remark. (a) According to the just stated definition not all finite strings of symbols of the language of Propositional Calculus are propositional forms. For instance, the expressions p, q, r, $\neg p$, $(p \wedge q)$, $(\neg p \Rightarrow r)$, $((p \wedge q) \vee (\neg p \Rightarrow r))$ can easily be recognized as propositional forms, while the expressions like $(p\neg q)$, $\neg pp \Rightarrow)(r\neg$ obviously fail to be propositional forms. Less obvious is the finding that neither the expressions $p \wedge q$, $\neg p \Rightarrow r$, $(p \wedge q) \vee (\neg p \Rightarrow r)$ are propositional forms although we are inclined to recognize them to be. In order to reconcile the above definition with our intuition and the usual practice, we accept the convention of omitting the outermost parentheses (which clearly are superfluous) in any propositional form. Thus we consider the expressions like $p \wedge q$, $\neg p \Rightarrow r$, $(p \wedge q) \vee (\neg p \Rightarrow r)$ as denoting the propositional forms $(p \wedge q)$, $(\neg p \Rightarrow r)$, $((p \wedge q) \vee (\neg p \Rightarrow r))$, respectively.

(b) We could completely manage without the parentheses using the *Polish notation.* In that case point 2° of the above definition would be modified as follows:

2* if $A, B \in \mathrm{VF}(P)$ then $\neg A,\ \wedge AB,\ \vee AB,\ \Rightarrow AB,\ \Leftrightarrow AB \in \mathrm{VF}(P)$
(if the strings A, B are propositional forms over the set P then so are the strings $\neg A$, $\wedge AB$, $\vee AB$, $\Rightarrow AB$ and $\Leftrightarrow AB$)

For instance, in Polish notation the propositional form $(p \wedge q) \vee (\neg p \Rightarrow r)$ would be written as

$$\vee \wedge pq \Rightarrow \neg pr$$

However cumbersome and hardly legible this expression may appear to us, it should be realized that from the point of view of a computer assisted processing this aspect is of almost no importance.

(c) In spite of the names we have attached to the logical connectives pointing to their intended role, they should be regarded as mere graphical symbols deprived of any meaning for the moment. They will only acquire their usual meaning later on, when we develop the semantics of Propositional Calculus.

(d) It should be noted that the signs A, B, C used to denote arbitrary propositional forms, the sign P and the expression $\mathrm{VF}(P)$ denoting the set of all propositional variables and the set of all propositional forms, respectively, etc., do not belong to the language of Propositional Calculus — they are symbols or expressions of certain metalanguage we use in the study of Propositional Calculus.

Let us turn reader's attention to the point that $\mathrm{VF}(P)$ is the *smallest* set satisfying conditions 1° an 2°. This inconspicuous requirement endows us with a powerful tool for proving facts about propositional forms, namely with the proof method by *induction on complexity*: In order to establish that all propositional forms have some property it is enough to show that the set of all propositional forms having this property satisfies the above conditions 1° and 2°.

Theorem. *Let $M \subseteq \mathrm{VF}(P)$ be any set of propositional forms satisfying the following two conditions:*

1° $P \subseteq M$
(every propositional variable $p \in P$ belongs to the set M)

2° *if $A, B \in M$ then $\neg A$, $A \wedge B$, $A \vee B$, $A \Rightarrow B$, $A \Leftrightarrow B \in M$*
(M is closed with respect to the formation of propositional forms by means of logical connectives)

Then $M = \mathrm{VF}(P)$, i.e., every propositional form over P belongs to M.

The reader should compare the induction on complexity with the usual method of induction, used in proving that certain property holds for all natural numbers: Since the set $\mathbb{N}$ of all natural numbers is the smallest set containing 0 and closed with respect to the successor operation $n \mapsto n+1$, in order to show that certain set $M \subseteq \mathbb{N}$ contains all natural numbers, i.e., $M = \mathbb{N}$, it is enough to show that $0 \in M$ and, for every $n \in M$ also $n+1 \in M$. In the induction on complexity the role of the number $0 \in \mathbb{N}$ is played by the propositional variables $p \in P$, and the role of the successor operation is played by the logical connectives. Already in this moment it could be anticipated that for the sake of induction proofs it would be desirable to reduce the number of logical connectives for which the step 2° has to be performed to some minimal list. We will return to this point in the next paragraph.

Interpretations, Truth Tables and Logical Equivalence

Next to syntax we will develop the *semantics* of Propositional Calculus. Let us recall that in Logic we take no account of the content of propositions, and the propositional forms are indeed deprived of any content. Nevertheless, we can still examine the situations under which they become true or false. These situations will be called interpretations or truth evaluation and they will represent the way of assigning however limited but still certain meaning to propositional forms.

We start by introducing the boolean algebraic operations on the two element set $\{0,1\}$ of the truth values 0 (*false*) and 1 (*true*), corresponding to the logical connectives and denoted by the same symbols. They are given by the following tables:

$\neg$	0	1
	1	0

$\wedge$	0	1
0	0	0
1	0	1

$\vee$	0	1
0	0	1
1	1	1

$\Rightarrow$	0	1
0	1	1
1	0	1

$\Leftrightarrow$	0	1
0	1	0
1	0	1

In Propositional Calculus an *intepretation* or a *truth evaluation* is any mapping $I\colon P \to \{0,1\}$, i.e., any assignment of truth values 0 or 1 to the propositional variables. Intuitively, such an interpretation represents a possible situation described in terms of the assignment of truth values to the propositional variables.

Every interpretation $I\colon P \to \{0,1\}$ will be extended to a mapping $I\colon \mathrm{VF}(P) \to \{0,1\}$, denoted by the same symbol and still called an interpretation or a truth evaluation, by means of the following recursive definition

$$I(\neg A) = \neg I(A) \qquad I(A \wedge B) = I(A) \wedge I(B) \qquad I(A \Rightarrow B) = I(A) \Rightarrow I(B)$$
$$I(A \vee B) = I(A) \vee I(B) \qquad I(A \Leftrightarrow B) = I(A) \Leftrightarrow I(B)$$

for any $A, B \in \mathrm{VF}(P)$, assuming that the values $I(A)$ and $I(B)$ have already been defined. Instead of $I(A) = 1$ we say that A is true or satisfied in the interpretation I; $I(A) = 0$ means that A is false in the interpretation I.

The reader should realize the following two facts:

- In each of the above equalities the signs $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$ denote the logical connectives on the left side while on the right side they denote the corresponding boolean operations on the set $\{0,1\}$.
- The equality symbol $=$ and the signs I, J, denoting arbitrary truth evaluations, belong to our metalanguage and not to the language of Propositional Calculus itself.

Just from this moment on, and by the virtue of the tables of the operations $\neg$, $\wedge$, $\vee$, $\Rightarrow$ and $\Leftrightarrow$ on the set $\{0,1\}$ of the truth values, the corresponding logical connectives can rightfully bear their names of *negation*, *conjunction*, *alternative* or *disjunction* (in nonexclusive sense), *implication* and *equivalence*, respectively.

It can be easily realized that the above recursive definition is redundant in some sense. It would be enough to describe the extension of the mapping $I\colon P \to \{0,1\}$ according to the negation $\neg$ and one (and anyone) of the binary connectives $\wedge$, $\vee$, $\Rightarrow$; then the remaining equalities would be satisfied, as well. In other words, a mapping $I\colon \mathrm{VF}(P) \to \{0,1\}$ is (an extension of) an interpretation if and only if it satisfies the equality $I(\neg A) = \neg I(A)$, and one (and anyone) of the equalities $I(A \wedge B) = I(A) \wedge I(B)$, $I(A \vee B) = I(A) \vee I(B)$, $I(A \Rightarrow B) = I(A) \Rightarrow I(B)$ for all $A, B \in \mathrm{VF}(P)$. Then it automatically satisfies the remaining equalities, as well. As a consequence, the notion of an interpretation (truth evaluation) could be defined in a more elegant way, as a mapping $I\colon \mathrm{VF}(P) \to \{0,1\}$ preserving the operations of the algebras $(\mathrm{VF}(P); \wedge, \vee, \neg)$, $(\{0,1\}; \wedge, \vee, \neg)$, i.e., as a *homomorphism* $I\colon (\mathrm{VF}(P); \wedge, \vee, \neg) \to (\{0,1\}; \wedge, \vee, \neg)$.

Let us make the just discussed point more precise. We call two propositional forms $A, B \in \mathrm{VF}(P)$ *logically equivalent* if $I(A) = I(B)$ for every interpretation $I\colon P \to \{0,1\}$; in that case we write $A \equiv B$. (It should be realized that the sign $\equiv$, similarly as the signs A, B, C, P, I, J or the expression $\mathrm{VF}(P)$, etc., does not belong to the symbols of the language of Propositional Calculus — it is a symbol of our metalanguage, again.) The reader is asked to verify that the relation of logical equivalence $\equiv$ is reflexive, symmetric and transitive, hence it is indeed an equivalence relation on the set $\mathrm{VF}(P)$.

It is known that any of the pairs $(\neg, \wedge)$, $(\neg, \vee)$, $(\neg, \Rightarrow)$ forms a *complete list of logical connectives*, i.e., any propositional form $A \in \mathrm{VF}(P)$ is logically equivalent to some propositional form A' containing the same propositional variables as A and involving just the logical connectives from one (and anyone) of the three pairs above.

Choosing the connectives $\neg$, $\wedge$ as the primitive ones, the remaining connectives could be introduced as abbreviations for the propositional forms on the right:

$$\begin{aligned} A \vee B &\equiv \neg(\neg A \wedge \neg B) \\ A \Rightarrow B &\equiv \neg(A \wedge \neg B) \\ A \Leftrightarrow B &\equiv \neg(A \wedge \neg B) \wedge \neg(\neg A \wedge B) \end{aligned}$$

Choosing $\neg$ and $\vee$ as primitive connectives we would have

$$\begin{aligned} A \wedge B &\equiv \neg(\neg A \vee \neg B) \\ A \Rightarrow B &\equiv \neg A \vee B \\ A \Leftrightarrow B &\equiv \neg(\neg(A \vee \neg B) \vee \neg(\neg A \vee B)) \end{aligned}$$

Finally, if our primitive connectives were $\neg$ and $\Rightarrow$, we would have

$$\begin{aligned} A \wedge B &\equiv \neg(A \Rightarrow \neg B) \\ A \vee B &\equiv \neg A \Rightarrow B \\ A \Leftrightarrow B &\equiv (A \Rightarrow \neg B) \Rightarrow \neg(\neg A \Rightarrow B) \end{aligned}$$

It follows that we could have used just the binary connective $\neg$ and just one (and anyone) from among the three binary connectives $\wedge$, $\vee$, $\Rightarrow$ in the recursive definition of the notion of propositional form in the previous paragraph; the forth binary connective $\Leftrightarrow$ becomes superfluous in any case.

Additionally, we will make use of the logical equivalences of associativity of the connectives $\wedge$ and $\vee$

$$(A \wedge B) \wedge C \equiv A \wedge (B \wedge C) \quad \text{and} \quad (A \vee B) \vee C \equiv A \vee (B \vee C)$$

for any propositional forms $A, B, C \in \mathrm{VF}(P)$. This allows us to omit the superfluous parenthesis in conjunctions and alternatives of an arbitrary finite number of propositional forms and write simply $A \wedge B \wedge C$, $A \vee B \vee C$, $A_1 \wedge \ldots \wedge A_m$, $B_1 \vee \ldots \vee B_n$, etc.

It is worthwhile to notice that we could manage with a single binary connective, namely the *Sheffer stroke* $|$ (the NAND operator) which can be expressed by means of $\neg$ and $\wedge$, or by means of $\neg$ and $\vee$ as follows

$$A|B \equiv \neg(A \wedge B) \equiv \neg A \vee \neg B$$

Conversely, the standard logical connectives $\neg$, $\wedge$ and $\vee$ can be expressed in terms of the Sheffer stroke as follows

$$\begin{aligned} \neg A &\equiv A|A \\ A \wedge B &\equiv (A|B)|(A|B) \\ A \vee B &\equiv (A|A)|(B|B) \end{aligned}$$

The task to find the corresponding expressions for $A \Rightarrow B$ and $A \Leftrightarrow B$ is left to the reader.

Another single logical connective capable to generate all the remaining ones is the NOR operator $\dagger$, also known as the *Peirce arrow* or *Quine dagger*, which is dual to the Sheffer stroke. In terms of $\neg$ and $\wedge$, or $\neg$ and $\vee$, respectively, it can be expressed as follows

$$A \dagger B \equiv \neg A \wedge \neg B \equiv \neg(A \vee B)$$

The reader is asked to express the usual logical connectives $\neg$, $\wedge$ and $\vee$, $\Rightarrow$ and $\Leftrightarrow$ in terms of the Quine dagger $\dagger$, and, at the same time to find the expressions for the Sheffer stroke in terms of the Quine dagger and vice versa.

Tautologies and Other Classes of Propositional Forms

Using the concept of interpretation we can single out several important classes of propositional forms. A propositional form $A \in \mathrm{VF}(P)$ is called

- a *tautology* if $I(A) = 1$ for every interpretation $I\colon P \to \{0,1\}$
- a *contradiction* if $I(A) = 0$ for every interpretation $I\colon P \to \{0,1\}$
- *satisfiable* if $I(A) = 1$ for at least one interpretation $I\colon P \to \{0,1\}$
- *refutable* if $I(A) = 0$ for at least one interpretation $I\colon P \to \{0,1\}$

There is a twofold duality between the four notions above: the inner duality

- A is a tautology if and only if $\neg A$ is a contradiction
- A is satisfiable if and only if $\neg A$ is refutable

and the outer duality

- A is a tautology if and only if A is not refutable
- A is a contradiction if and only if A is not satisfiable

It can be easily seen that, for any propositional forms A, B, we have $A \equiv B$ if and only if the propositional form $A \Leftrightarrow B$ is a tautology.

The question whether an arbitrary propositional form A belongs to any of the four classes defined above can be decided algorithmically using the method of *truth tables*, evaluating the truth values $I(A)$ for all the interpretations $I\colon P \to \{0,1\}$. In view of the fact that, for an infinite set P, there are infinitely many such interpretations, it is important that to that end it is enough to deal just with finitely many of them.

Theorem. *Let $A \in \mathrm{VF}(P)$ be any propositional form such that all the propositional variables occurring in A are included in the list $p_1, \dots, p_n$. Then $I(A) = J(A)$ for any truth evaluations $I, J\colon P \to \{0,1\}$ such that $I(p_k) = J(p_k)$ for each $k = 1, \dots, n$.*

In other words, the value $I(A)$ of a truth evaluation I on a propositional form A depends on the values of I on the finite set of propositional variables occurring in A, only. However obvious and intuitively clear this fact may appear, we nonetheless prove it, mainly in order to illustrate the proof method by induction on complexity.

Demonstration. Denoting $Q = \{p_1, \dots, p_n\}$ and

$$M = \{A \in \mathrm{VF}(Q)\colon I(A) = J(A)\}$$

we are to show that $M = \mathrm{VF}(Q)$. Since I and J coincide on the set Q, we have $Q \subseteq M$, which is the initial induction step 1°. In order to verify the induction step 2°, assume that $A, B \in M$, i.e., $A, B \in \mathrm{VF}(Q)$, and $I(A) = J(A)$ as well as $I(B) = J(B)$. Then, as both I, J preserve the logical connectives,

$$I(\neg A) = \neg I(A) = \neg J(A) = J(\neg A)$$
$$I(A \wedge B) = I(A) \wedge I(B) = J(A) \wedge J(B) = J(A \wedge B)$$

hence both $\neg A, A \wedge B \in M$. Similarly, we could show that $A \vee B, A \Rightarrow B, A \Leftrightarrow B \in M$, too. However, in view of our previous accounts, it is clear that the induction step 2° for the connectives $\vee$, $\Rightarrow$ and $\Leftrightarrow$ is not necessary to perform.

Example. Using the truth table method, it can be easily shown that the following propositional form

$$(p \Rightarrow (q \Rightarrow r)) \Leftrightarrow ((p \wedge q) \Rightarrow r)$$

is a tautology. Denoting by L the propositional form $p \Rightarrow (q \Rightarrow r)$ and by R the propositional form $(p \wedge q) \Rightarrow r$, we have

p	q	r	$q \Rightarrow r$	L	$p \wedge q$	R	$L \Leftrightarrow R$
1	1	1	1	1	1	1	1
1	1	0	0	0	1	0	1
1	0	1	1	1	0	1	1
0	1	1	1	1	0	1	1
1	0	0	1	1	0	1	1
0	1	0	0	1	0	1	1
0	0	1	1	1	0	1	1
0	0	0	1	1	0	1	1

As a consequence,

$$A \Rightarrow (B \Rightarrow C) \equiv (A \wedge B) \Rightarrow C$$

for any propositional forms A, B, C.

More important and interesting than filling in mechanically the above truth table it is to realize what kind of logical law, called the *Law of Exportation*, is expressed by this tautology or by the above logical equivalence. The left hand form $A \Rightarrow (B \Rightarrow C)$ states that "if A, then B implies C". The right hand form $(A \wedge B) \Rightarrow C$ states that "A and B jointly imply C". These two forms of statements are always equivalent: going from the left to the right it is possible to join the two assumptions A, B to a single assumption $A \wedge B$; going from the right to the left it is possible to divide the assumption $A \wedge B$ into its constituents A and B and apply them consecutively one after the other.

Exercise. A propositional form is called an *elementary conjunction* if it has the shape $B_1 \wedge \ldots \wedge B_m$ where each of the forms B_i is either a propositional variable or a negation of some propositional variable. A propositional form is called a *disjunctive normal form* if it has the shape $C_1 \vee \ldots \vee C_k$ where each of the forms C_j is an elementary conjunction. Show that every propositional form $A \in \mathrm{VF}(p_1, \ldots, p_n)$ is logically equivalent to some disjunctive normal form $A' \in \mathrm{VF}(p_1, \ldots, p_n)$. To this end design an algorithmic method how to obtain the disjunctive normal form $A' \equiv A$ from the truth table of the form A.

Similarly, define the dual notions of an *elementary disjunction* and of a *conjunctive normal form* and show that every propositional form is logically equivalent to some conjunctive normal form.

Exercise. Let $A \in \mathrm{VF}(p_1, \ldots, p_n)$ be a propositional form in propositional variables $p_1, \ldots, p_n$ and $B_1, \ldots, B_n \in \mathrm{VF}(P)$ be arbitrary propositional forms. We denote by $A(B_1, \ldots, B_n)$ the propositional form obtained by substituting the forms $B_1, \ldots, B_n$ into the form A in places of the variables $p_1, \ldots, p_n$, respectively. For instance, if A is the form $(p \wedge \neg q) \Rightarrow (q \vee r)$ in propositional variables p, q, r and B, C, D are the propositional forms $r \vee s$, $p \Rightarrow \neg r$, q, respectively, then $A(B, C, D)$ denotes the form

$$((r \vee s) \wedge \neg(p \Rightarrow \neg r)) \Rightarrow ((p \Rightarrow \neg r) \vee q)$$

(a) Demonstrate that if A is a tautology (contradiction) then $A(B_1, \ldots, B_n)$ is also a tautology (contradiction) for any $B_1, \ldots, B_n$.

(b) Give examples of a satisfiable (refutable) form A and of forms $B_1, \ldots, B_n$ such that $A(B_1, \ldots, B_n)$ is not satisfiable (refutable).

Theories in Propositional Calculus

In common language the word *theory* usually refers to some interconnected system of knowledge, consisting of statements about certain topic and including also a methodology of obtaining and verifying or refuting these statements. The statements or propositions forming the "body of knowledge" of the theory could have been obtained in various ways: some of them may express certain empirical facts established by observation or experiments, some of them may be a part of common beliefs, tradition or cultural heritage, some of them may be mere hypotheses to be verified or refuted in the future, and, finally, some of them may be derived from any of the previously mentioned ones as their logical consequences.

Following the leading intention of logic, we will ignore the content, methodology and the overall character of a theory, we will neither distinguish which of its postulates are true or false, which are firmly established and which are mere hypotheses, nor take care of the way how all that happened. We will bring to the focus just a single aspect of all such theories, namely the structure of logical inference, i.e., the way new statements necessarily follow or can be derived from those made into the departing postulates or axioms of the particular theory.

Accordingly, a *propositional theory* or simply a *theory* is any set $T \subseteq \mathrm{VF}(P)$ of propositional forms; its elements $A \in T$ are called the *specific axioms* or just the *axioms* of T. We warn the readers not to take this definition word for word, not even within the framework of Propositional Calculus, let alone when speaking about a broader perspective. It should rather be understood as stating that, within Propositional Calculus, a theory *is given* or *uniquely determined* by the set of its specific axioms. Propositional Calculus will take care of the rest, i.e., of the structure of logical inference, which is the same for all the theories.

An interpretation $I\colon \mathrm{VF}(P) \to \{0,1\}$ is called an *interpretation of the theory* T if $I(A) = 1$ for each $A \in T$, i.e., if all the axioms of T are true in the interpretation I. Intuitively, an interpretation of the theory T represents a situation in which all the axioms of the theory T, hence T itself, are satisfied.

A propositional form B is a *logical consequence* of the axioms of a theory T or just a *logical consequence* of T if $I(B) = 1$ for every interpretation I of the theory T. Alternatively we say that B *is true* or *valid* or *satisfied* in T, or that T *entails* B. In symbols we write $T \vDash B$. Intuitively, $T \vDash B$ means that, in every possible situation in which all the axioms of the theory T are satisfied, B is satisfied as well.

Instead of $\emptyset \vDash B$ we write just $\vDash B$; it means that B is true under every interpretation $I\colon P \to \{0,1\}$, in other words, B is a tautology.

As it follows from the theorem below, the question whether $T \vDash B$ can be algorithmically decided using truth tables, for any theory T with just finitely many specific axioms and each propositional form $B \in \mathrm{VF}(P)$.

Theorem. *Let $T = \{A_1, \ldots, A_n\}$ be a theory with finitely many specific axioms and $B \in \mathrm{VF}(P)$. Then $T \vDash B$ if and only if the propositional form $(A_1 \wedge \ldots \wedge A_n) \Rightarrow B$ is a tautology.*

Demonstration. Assume that $T \models B$. Let $I\colon \mathrm{VF}(P) \to \{0,1\}$ be any interpretation. Then either $I(A_k) = 0$ for at least one $k = 1, \dots, n$, or $I(A_k) = 1$ for each $k = 1, \dots, n$. In the first case $I(A_1 \wedge \dots \wedge A_n) = 0$, therefore,

$$I((A_1 \wedge \dots \wedge A_n) \Rightarrow B) = 1$$

In the second case I is an interpretation of the theory T, hence $I(B) = 1$ since $T \models B$. Then

$$I((A_1 \wedge \dots \wedge A_n) \Rightarrow B) = 1$$

again. Thus $(A_1 \wedge \dots \wedge A_n) \Rightarrow B$ is indeed a tautology.

Conversely, assume that $(A_1 \wedge \dots \wedge A_n) \Rightarrow B$ is a tautology, i.e., it is true in every interpretation I. If I is an interpretation of T, then $I(A_1 \wedge \dots \wedge A_n) = 1$. Thus

$$I((A_1 \wedge \dots \wedge A_n) \Rightarrow B) = 1$$

can happen only if $I(B) = 1$, too. It follows that $T \models B$.

In general, however, T may have infinitely many specific axioms. Even in that case, in order to show that B is not a logical consequence of T, i.e., $T \not\models B$, it is enough to find a single interpretation I of T such that $I(B) = 0$. If this is the case, then we say that (the validity of) B in T was refuted by a counterexample. However, in order to confirm that $T \models B$, the definition requires of us to determine the truth value $I(B)$ for infinitely many interpretations of T, which seems to be an unrealizable task.

In mathematics, however, the usual way how to establish the validity of some statement within some theory is by *proving it from the axioms of the theory* and not by examining all the possible situations in which these axioms are true and checking the validity of the statement in each of these situations. Also in Propositional Calculus we will develop the syntactic concepts of proof and provability with the aim to get in grasp with the semantic concept of validity or truth by means of them.

Axiomatization of Propositional Calculus

In order to have a brief and concise axiomatization of Propositional Calculus we will proceed as if the set $\mathrm{VF}(P)$ of all propositional forms were built of the propositional variables by means of the logical connectives $\neg$ and $\Rightarrow$, only. Thus the remaining logical connectives are considered as certain abbreviations displayed in the previous paragraph. An alternative axiomatization using the logical connectives $\neg$, $\wedge$, $\vee$ and $\Rightarrow$ can be found in the Appendix to this Chapter.

Logical axioms. (4 axiom schemes)
For any propositional forms A, B, C, the following propositional forms are logical axioms:

(LAx 1) $A \Rightarrow (B \Rightarrow A)$
(LAx 2) $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$
(LAx 3) $(A \Rightarrow B) \Rightarrow ((A \Rightarrow \neg B) \Rightarrow \neg A)$
(LAx 4) $\neg\neg A \Rightarrow A$

Additionally, we have a single deduction rule or rule of inference:

Deduction rule MODUS PONENS

(MP) $\dfrac{A,\ A \Rightarrow B}{B}$ (from A and $A \Rightarrow B$ infer B)

Exercise. (a) Show that all the logical axioms are tautologies and explain their intuitive meaning.
(b) Show that the inference rule Modus Ponens is correct in the following sense:

If $I\colon \mathrm{VF}(P) \to \{0,1\}$ is any interpretation and $A, B \in \mathrm{VF}(P)$ are propositional forms such that $I(A) = I(A \Rightarrow B) = 1$, then $I(B) = 1$, as well.

A *proof* in the theory $T \subseteq \mathrm{VF}(P)$ is a finite sequence $A_0, A_1, \ldots, A_n$ of propositional forms such that every item A_k is either a logical axiom, or a specific axiom of the theory T (i.e. $A_k \in T$), or it follows from the previous items by the rule (MP) (i.e., there are $i, j < k$ such that A_j has the form $A_i \Rightarrow A_k$).

A propositional form B is *provable* in a theory T if there is a proof $A_0, A_1, \ldots, A_n$ in T such that its last item A_n coincides with B. In symbols, $T \vdash B$. Instead of $\emptyset \vdash B$ we write just $\vdash B$; it means that B is provable from the logical axioms, only.

Remark. The above axiomatization of Propositional Calculus is by far not the only possible one. As already mentioned, an alternative axiomatization can be found in the Appendix. Both of these axiomatizations contain infinitely many axioms (listed in form of finitely many axiom schemes) and a single rule of inference. Such axiomatizations are referred to as *Hilbert style axiomatizations* featured by "many" logical axioms and "few" rules of inference. On the other hand, the *Gentzen style axiomatizations* contain "many" rules of inference and just "few" logical axioms (or even none, replacing a logical axiom A by the deduction rule $\frac{}{A}$ with meaning *derive A out of nothing*). In general, Hilbert style axiomatizations are better suited for the description, study and analysis of the formal logical system itself, while Gentzen style axiomatizations are more effective in applications like logical programming or automatic theorem proving. However, as far as they serve as axiomatizations of the classical Propositional Calculus, they are all equivalent in the sense that they produce the same family of provable forms.

Exercise. Show that, for any propositional forms A, B, the following propositional forms are tautologies, and that they all are provable just from the logical axioms:
(a) $A \Rightarrow A$
(b) $A \Rightarrow \neg\neg A$
(c) $\neg A \Rightarrow (A \Rightarrow B)$
(d) $(\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B)$
(e) $(A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)$
(f) $(A \Rightarrow (\neg B \Rightarrow \neg(A \Rightarrow B)))$
(g) $(A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B)$
(h) $(\neg A \Rightarrow A) \Rightarrow A$

As an example (a rather deterring one) we just show that for every propositional form A the form in (a) is provable from the logical axioms.

1. $(A \Rightarrow ((A \Rightarrow A) \Rightarrow A)) \Rightarrow ((A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A))$
   (LAx 2), taking A for both A and C and $A \Rightarrow A$ for B
2. $A \Rightarrow ((A \Rightarrow A) \Rightarrow A)$
   (LAx 1), taking A for A and $A \Rightarrow A$ for B
3. $(A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A)$
   follows from 1. and 2. by (MP)
4. $A \Rightarrow (A \Rightarrow A)$
   (LAx 1), taking A for both A and B
5. $A \Rightarrow A$
   follows from 3. and 4. by (MP)

Exercise. Sow that the axiom schemes (LAx 3) and (LAx 4) can be replaced by a single axiom scheme

(LAx 5) $(\neg A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow \neg B) \Rightarrow A)$

To this end show that every instance of the scheme (LAx 5) is provable from some instances of the schemes (LAx 1), (LAx 2), (LAx 3), (LAx 4), and vice versa, all instances of the schemes (LAx 3), (LAx 4) are provable from some instances of the schemes (LAx 1), (LAx 2), (LAx 5).

The Soundness Theorem

Having introduced the axiomatization of Propositional Calculus we are facing the task to establish that it is *sound* or *correct* in the following sense: For every theory $T \subseteq \mathrm{VF}(P)$, all the propositional forms provable in T are satisfied in T. Otherwise it could happen that for some propositional form B provable in T it would be possible to find an interpretation I of T such that $I(B) = 0$. Such an I would represent a situation in which all the axioms of T were satisfied, nevertheless, B were false. Thus we could be able to prove false conclusions from the axioms of T which would be a disaster witnessing a collapse of our axiomatization. Therefore it is of crucial importance that we have the following

Soundness Theorem. *Let $T \subseteq \mathrm{VF}(P)$ be a theory. Then, for every propositional form $B \in \mathrm{VF}(P)$, if $T \vdash B$ then $T \models B$.*

Demonstration. Let $T \vdash B$ and $A_0, A_1, \ldots, A_n$ be a proof of B in T. We will show that $I(A_k) = 1$ for every interpretation I of the theory T and *each* $k \leq n$. Then, of course, $I(B) = 1$, since B is A_n. Each A_k is either a logical axiom, in which case $I(A_k) = 1$ for every interpretation I, or a specific axiom of T, in which case $I(A_k) = 1$ as I is an interpretation of T, or A_k follows from some previous items A_i, A_j by (MP). Assuming that we already have proved that $I(A_i) = I(A_j) = 1$, we can conclude $I(A_k) = 1$, too, since, as we already have noted, the rule (MP) is correct.

Remark. Let us turn the reader's attention to the fact that — however simple and transparent the above argument might appear — it contains a kind of vicious circle. In the demonstration of the Soundness Theorem we have been using logical deduction and inference within the natural language extended by some fairly simple mathematical notation. Thus we have used in an informal way the same logical means the soundness of which we wanted to establish within the formalized Propositional Calculus. Strictly speaking, the formal counterpart of the informal logical means we have been using goes even beyond Propositional Calculus: since our arguments contain some quantification, they interfere already with the Predicate Calculus. It is important to realize that we are unable to prove the Soundness Theorem out of nothing, without assuming some minimal logical fragment of natural language as granted. Thus what we have achieved is nothing more and nothing less than the understanding and realization that our formalized axiomatization of Propositional Calculus is in good accord with the logical structure of deduction and inference within our natural language.

Later on we will also establish the converse of the Soundness Theorem.

Completeness Theorem. *Let $T \subseteq \mathrm{VF}(P)$ be a theory. Then, for every propositional form $B \in \mathrm{VF}(P)$, if $T \vDash B$ then $T \vdash B$.*

Remark. It is illuminating to compare the status of the Completeness Theorem with that of the Soundness Theorem. As we have seen, the demonstration of the Soundness Theorem was fairly simple. On the other hand, as we shall see later on, the demonstration of the Completeness Theorem will be considerably more involved. While the failure of the Soundness Theorem would cause a collapse of our axiomatization of Propositional Calculus, the consequences of a possible failure of the Completeness Theorem would be, at least at first glance, less dramatic: It would just mean that our axiomatization of Propositional Calculus is not powerful enough and we should look for some additional logical axioms and/or deduction rules extending our original list in order to achieve its completeness. Then, however, we would have to face a more delicate question: Is it at all possible to achieve completeness in our axiomatization without destroying its soundness? Namely, the Soundness Theorem and the Completeness Theorem together answer this question affirmatively and guarantee that the relation between the syntax and semantics of Propositional Calculus is carefully balanced.

Later on, when dealing with an analogous issue for Predicate Calculus, we will quote an example of its certain fairly natural fragment not admitting any axiomatization satisfying both the Soundness and the Completeness Theorem.

The Deduction Theorem and Its Corollaries

On the way to the demonstration of the Completeness Theorem we are going to state a handful of results which are of independent interest in their own right. In their demonstrations we will use the notation $A \approx B$, expressing that the characters A and B denote the same propositional form. The symbol $\approx$ belongs to our metalanguage and not to the language of Propositional Calculus itself, similarly as the symbols A, B, P, VF, I, $\equiv$, etc.

Deduction Theorem. *Let $T \subseteq \mathrm{VF}(P)$ be a theory and $A, B \in \mathrm{VF}(P)$ be propositional forms. Then $T \vdash A \Rightarrow B$ if and only if $T \cup \{A\} \vdash B$.*

Demonstration. Let $T \vdash A \Rightarrow B$. Then the more $T \cup \{A\} \vdash A \Rightarrow B$. Obviously, $T \cup \{A\} \vdash A$, from which we get $T \cup \{A\} \vdash B$ by (MP). Namely, if $C_0, C_1, \ldots, C_n$ is a proof of $A \Rightarrow B$ in $T \cup \{A\}$, then $C_0, C_1, \ldots, C_n, A, B$ is a proof of B in $T \cup \{A\}$.

Conversely, let $T \cup \{A\} \vdash B$. First we take care of the following two trivial cases:

(a) B is a logical axiom or $B \in T$. Then B, $B \Rightarrow (A \Rightarrow B)$ (LAx 1), $A \Rightarrow B$ is a proof of $A \Rightarrow B$ in T.

(b) $B \approx A$. Then $\vdash A \Rightarrow A$ (Exercise (a)), hence the more $T \vdash A \Rightarrow A$.

Otherwise there must be a proof $B_0, B_1, \ldots, B_n$ of B in the theory $T \cup \{A\}$ such that $n \geq 2$ and B_n (i.e. B) follows from some previous items of this sequence by (MP). We will proceed by induction according to n. To this end we assume that the needed conclusion is valid for all proofs $C_0, C_1, \ldots, C_m$ in $T \cup \{A\}$, where $m < n$. Let $j, k < n$ be such that $B_j \approx (B_k \Rightarrow B_n)$. Then both $B_0, \ldots, B_j$ and $B_0, \ldots, B_k$ are proofs in $T \cup \{A\}$. By the induction assumption we have $T \vdash A \Rightarrow B_j$, i.e., $T \vdash A \Rightarrow (B_k \Rightarrow B_n)$, as well as $T \vdash A \Rightarrow B_k$. Then

$$(A \Rightarrow (B_k \Rightarrow B_n)) \Rightarrow ((A \Rightarrow B_k) \Rightarrow (A \Rightarrow B_n))$$

is (LAx 2), and by (MP) we consecutively get

$$T \vdash (A \Rightarrow B_k) \Rightarrow (A \Rightarrow B_n)$$
$$T \vdash A \Rightarrow B_n$$

i.e., $T \vdash A \Rightarrow B$.

The reader should notice that it is the "harder" implication

If $T \cup \{A\} \vdash B$ then $T \vdash A \Rightarrow B$

which is frequently used in mathematical proofs as well as in many deductive arguments elsewhere. A typical direct proof of the implication $A \Rightarrow B$ out of a list (theory) T of assumptions (axioms) starts with the "ritual" formulation: *"Let A"*, or *"Assume that A"*. This is nothing else than extending the axiom list T by a new axiom A. We continue by a sequence of statements $C_1, \ldots, C_n$ formed according to some deductive rules and finish once we succeed to arrive at the final term B. However, strictly speaking, what we have produced that way is a proof of B within the theory $T \cup \{A\}$ and not a proof of the implication $A \Rightarrow B$ in T as we claim. The Deduction Theorem shows that this natural method of argumentation is legitimate within Propositional Calculus, justifying our claim.

Another way of proving a statement out of some list of assumptions is the *proof by contradiction.* Instead of proving A in T directly, we produce a contradiction with the axioms of T out of the negation of A. Also this method is legitimate in Propositional Calculus.

A theory T is called *contradictory* or *inconsistent* if there exists some propositional form A such that both $T \vdash A$ and $T \vdash \neg A$. Otherwise, T is called *consistent.* From the Exercise (c) it follows that every propositional form B is provable in an inconsistent theory T.

Corollary on Proof by Contradiction. *Let $T \subseteq \mathrm{VF}(P)$ be a theory and $A \in \mathrm{VF}(P)$ be a propositional form. Then $T \vdash A$ if and only if the theory $T \cup \{\neg A\}$ is contradictory (inconsistent).*

Demonstration. Let $T \vdash A$. The more $T \cup \{\neg A\} \vdash A$. Since, clearly, $T \cup \{\neg A\} \vdash \neg A$, the theory $T \cup \{\neg A\}$ is contradictory.

Conversely, let the theory $T \cup \{\neg A\}$ be contradictory. Then every propositional form is provable in this theory; in particular, $T \cup \{\neg A\} \vdash A$. Then $T \vdash \neg A \Rightarrow A$ by the Deduction Theorem. According to Exercise (h), $\vdash (\neg A \Rightarrow A) \Rightarrow A$, and the more $T \vdash (\neg A \Rightarrow A) \Rightarrow A$. Using (MP) we get $T \vdash A$.

Sometimes we are unable to find a proof of a statement B in a theory T, however, we are able to prove B under some additional assumption A in one way, and in another way under the opposite assumption $\neg A$. Then, all the same, it follows that B is provable in T. This way of argumentation is legitimate in Propositional Calculus, as well.

Corollary on Proof by Distinct Cases. *Let $T \subseteq \mathrm{VF}(P)$ be a theory and $A, B \in \mathrm{VF}(P)$ be propositional forms. Then $T \cup \{A\} \vdash B$ and $T \cup \{\neg A\} \vdash B$ if and only if $T \vdash B$.*

Demonstration. Assume that $T \cup \{A\} \vdash B$ and $T \cup \{\neg A\} \vdash B$. According to the Deduction Theorem it follows $T \vdash A \Rightarrow B$ and $T \vdash \neg A \Rightarrow B$. By Exercise (g) we have

$$\vdash (A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B)$$

and applying (MP) twice we get $T \vdash B$.

Conversely, let $T \vdash B$. Then, trivially, $T \cup \{A\} \vdash B$, as well as $T \cup \{\neg A\} \vdash B$.

Exercise. Let $T \subseteq \mathrm{VF}(P)$ be a theory and $A_1, \ldots, A_n, B \in \mathrm{VF}(P)$ be propositional forms such that $T \vdash A_1 \vee \ldots \vee A_n$. Show that $T \vdash B$ if and only $T \cup \{A_i\} \vdash B$ for each $i = 1, \ldots, n$.

The Completeness Theorem

We start with a technical lemma. Given any interpretation $I\colon \mathrm{VF}(P) \to \{0, 1\}$ and a propositional form $A \in \mathrm{VF}(P)$ we denote

$$A^I \approx \begin{cases} A & \text{if } I(A) = 1 \\ \neg A & \text{if } I(A) = 0 \end{cases}$$

In other words, A^I is namely that member of the couple A, $\neg A$ which is true in I, i.e., $I(A^I) = 1$.

Lemma on Interpretation. [A. Church] *Let* $p_1, \ldots, p_n \in P$ *and* $A \in \mathrm{VF}(p_1, \ldots, p_n)$. *Then for any interpretation* $I\colon \mathrm{VF}(P) \to \{0,1\}$ *we have*

$$\{p_1^I, \ldots, p_n^I\} \vdash A^I$$

Demonstration. By induction on complexity of A:

(a) If $A \approx p \in P$, then the statement means that $\{p\} \vdash p$, if $I(p) = 1$, or $\{\neg p\} \vdash \neg p$, if $I(p) = 0$. In both cases we get the needed conclusion.

(b) Let $A \approx \neg B$ and our conclusion be true for B. Then $B \in \mathrm{VF}(p_1, \ldots, p_n)$.

If $I(A) = 1$, then $I(B) = 0$ and $A^I \approx A \approx \neg B \approx B^I$. By the assumption, $\{p_1^I, \ldots, p_n^I\} \vdash B^I$, i.e., $\{p_1^I, \ldots, p_n^I\} \vdash A^I$.

If $I(A) = 0$, then $I(B) = 1$, $B^I \approx B$ and $A^I \approx \neg A \approx \neg\neg B$. By the assumption $\{p_1^I, \ldots, p_n^I\} \vdash B^I$, i.e., $\{p_1^I, \ldots, p_n^I\} \vdash B$. According to Exercise (b) we have $\vdash B \Rightarrow \neg\neg B$, and by (MP) we get $\{p_1^I, \ldots, p_n^I\} \vdash \neg\neg B$, i.e., $\{p_1^I, \ldots, p_n^I\} \vdash A^I$.

(c) Let $A \approx (B \Rightarrow C)$ and for B, C the conclusion is true. Then $B, C \in \mathrm{VF}(p_1, \ldots, p_n)$. We distinguish three cases:

1. $I(B) = 0$. Then $I(A) = I(B \Rightarrow C) = 1$, i.e., $A^I \approx A$. Further, $B^I \approx \neg B$, hence, by the induction assumption, $\{p_1^I, \ldots, p_n^I\} \vdash \neg B$. According to Exercise (c) we have $\vdash \neg B \Rightarrow (B \Rightarrow C)$ and by (MP) we get $\{p_1^I, \ldots, p_n^I\} \vdash B \Rightarrow C$, i.e., $\{p_1^I, \ldots, p_n^I\} \vdash A^I$.

2. $I(C) = 1$. Then $C^I \approx C$ and $I(A) = I(B \Rightarrow C) = 1$, hence $A^I \approx A$. By the induction assumption, $\{p_1^I, \ldots, p_n^I\} \vdash C$. (LAx 1) gives $\vdash C \Rightarrow (B \Rightarrow C)$, and by (MP) we get $\{p_1^I, \ldots, p_n^I\} \vdash B \Rightarrow C$, i.e., $\{p_1^I, \ldots, p_n^I\} \vdash A^I$.

3. $I(B) = 1$, $I(C) = 0$. Then $B^I \approx B$, $C^I \approx \neg C$ and $I(A) = I(B \Rightarrow C) = 0$, hence $A^I \approx \neg A$. By the induction assumption, $\{p_1^I, \ldots, p_n^I\} \vdash B$ and $\{p_1^I, \ldots, p_n^I\} \vdash \neg C$. Exercise (f) gives $\vdash B \Rightarrow (\neg C \Rightarrow \neg(B \Rightarrow C))$. Using (MP) twice we get $\{p_1^I, \ldots, p_n^I\} \vdash \neg(B \Rightarrow C)$, i.e., $\{p_1^I, \ldots, p_n^I\} \vdash A^I$.

Exercise. Let $Q = \{p_1, \ldots, p_n\} \subseteq P$ be a finite set of propositional variables and $A \in \mathrm{VF}(Q)$. Let

$$\mathrm{TE}(A) = \bigl\{I\colon Q \to \{0,1\} \colon I(A) = 1\bigr\} = \{I_1, \ldots, I_m\}$$

denote the set of all truth evaluations I on the set of propositional variables Q such that A is true in I. Obviously, $m \leq 2^n$. For each $I \in \mathrm{TE}(A)$ we denote by

$$C_I = p_1^I \wedge \ldots \wedge p_n^I$$

the elementary conjunction corresponding to I. Finally, we put

$$A' = C_{I_1} \vee \ldots \vee C_{I_m}$$

Give reasons for the claim that $A' \in \mathrm{VF}(Q)$ is a disjunctive normal form logically equivalent to A (cf. Exercise…)

A special case of the Completeness Theorem deals with the provability of tautologies.

Completeness Theorem for Tautologies. [E. Post] *For every propositional form $A \in \mathrm{VF}(P)$, we have $\models A$ if and only if $\vdash A$; in other words, A is a tautology if and only if A is provable just from the logical axioms.*

Demonstration. We just show that every tautology is provable from the logical axioms; the converse follows from the Soundness Theorem.

Let $A \in \mathrm{VF}(p_1, \ldots, p_n)$. Since A is a tautology, $I(A) = 1$ and $A^I \approx A$ for every truth evaluation $I\colon \{p_1, \ldots, p_n\} \to \{0, 1\}$. By the Interpretation Lemma,

$$\{p_1^I, \ldots, p_n^I\} \vdash A$$

For any truth evaluation $J\colon \{p_1, \ldots, p_{n-1}\} \to \{0, 1\}$, both possibilities $I_1(p_n) = 1$, $I_2(p_n) = 0$ jointly with the condition $I_1(p_k) = I_2(p_k) = J(p_k)$, for $k < n$, produce interpretations $I_1, I_2\colon \{p_1, \ldots, p_n\} \to \{0, 1\}$. Therefore, both

$$\{p_1^J \ldots, p_{n-1}^J, p_n\} \vdash A$$
$$\{p_1^J \ldots, p_{n-1}^J, \neg p_n\} \vdash A$$

According to Corollary on Proof by Distinct Cases this implies

$$\{p_1^J, \ldots, p_{n-1}^J\} \vdash A$$

Repeating this procedure we finally get $\vdash A$.

A theory $T \subseteq \mathrm{VF}(P)$ is called *complete* if it is consistent and for every propositional form $A \in \mathrm{VF}(P)$ we have $T \vdash A$ or $T \vdash \neg A$. In other words, T is complete if and only if for every propositional form A exactly one of the two possibilities $T \vdash A$, $T \vdash \neg A$ takes place.

Next we show an alternative version of the Completeness Theorem.

Completeness Theorem. [Alternative version] *Every consistent theory $T \subseteq \mathrm{VF}(P)$ has an interpretation.*

The reader is asked to realize that also the other way round, if a theory has an interpretation then it is necessarily consistent; in other words, a contradictory theory has no interpretation. (This is the alternative version of the Soundness Theorem.)

Demonstration. Any interpretation I of a consistent theory T has to satisfy

$$I(A) = \begin{cases} 1 & \text{if } T \vdash A \\ 0 & \text{if } T \vdash \neg A \end{cases}$$

Since T is consistent, $T \vdash A$ and $T \vdash \neg A$ cannot happen at once for any $A \in \mathrm{VF}(P)$. On the other hand, unless T is complete, we cannot guarantee that we always have either $T \vdash A$ or $T \vdash \neg A$, i.e., the value $I(A)$ need not be defined for every $A \in \mathrm{VF}(P)$. However, if T is *complete* then the above casework defines an interpretation of T, indeed. In other words, a complete theory T has exactly one interpretation.

In the general case, since the set $\mathrm{VF}(P)$ of all propositional forms is countable, it allows for some enumeration $\mathrm{VF}(P) = \{A_0, A_1, \ldots, A_n, \ldots\}$. Now we define a sequence of theories $T_0 \subseteq T_1 \subseteq \ldots \subseteq T_n \subseteq T_{n+1} \subseteq \ldots$ recursively:

$$T_0 = T \quad \text{and} \quad T_{n+1} = \begin{cases} T_n \cup \{A_n\} & \text{if } T_n \cup \{A_n\} \text{ is consistent} \\ T_n \cup \{\neg A_n\} & \text{if } T_n \cup \{A_n\} \text{ is contradictory} \end{cases}$$

Obviously, $T_n \subseteq T_{n+1}$ for each n. Let us show by induction on n that every T_n is a consistent theory. $T_0 = T$ is consistent by the initial assumption. Assuming that T_n is consistent, T_{n+1} could be inconsistent only in case that both the theories $T_n \cup \{A_n\}$, $T_n \cup \{\neg A_n\}$ were contradictory. By the Corollary on Proof by Contradiction this would mean that both $T_n \vdash \neg A_n$ and $T_n \vdash A_n$. However, this is impossible, as T_n is consistent.

Next we show that $\widehat{T} = \bigcup_{n\in\mathbb{N}} T_n$ is a complete theory. It is easy to realize that $\widehat{T}$ is consistent. Indeed, if $\widehat{T}$ were inconsistent then already some of the theories T_n would inconsistent as well (this is left to the reader as an exercise — see also the proof of the Compactness Theorem). It remains to show that, for each n, either $\widehat{T} \vdash A_n$ or $\widehat{T} \vdash \neg A_n$. This is equivalent to showing that $\widehat{T} \nvdash \neg A_n$ implies $\widehat{T} \vdash A_n$. If $\widehat{T} \nvdash \neg A_n$ then $\widehat{T} \cup \{A_n\}$ is consistent, and $T_n \cup \{A_n\} \subseteq \widehat{T} \cup \{A_n\}$ is consistent, as well. Then $A_n \in T_{n+1}$, hence $T_{n+1} \vdash A_n$, and, since $T_{n+1} \subseteq \widehat{T}$, also $\widehat{T} \vdash A_n$.

Thus the unique interpretation I of the complete theory $\widehat{T}$ is an interpretation of T, as well.

Remark. The reader should notice that the above casework is not necessarily the only way how the sequence of theories $(T_n)_{n\in\mathbb{N}}$ extending T, leading to a complete theory $\widehat{T} = \bigcup T_n$, and the interpretation I could be defined. In each step when neither $T_n \vdash A_n$ nor $T_n \vdash \neg A_n$, we are free to choose either $T_{n+1} = T_n \cup \{A_n\}$ or $T_{n+1} = T_n \cup \{\neg A_n\}$.

Exercise. Let $I\colon P \to \{0,1\}$ be any truth evaluation. Let us denote

$$\mathrm{Th}(I) = \left\{p^I : p \in P\right\} = \{p \in P \colon I(p) = 1\} \cup \{\neg p \colon p \in P,\ I(p) = 0\}$$

the *theory of* I. Demonstrate the following facts:

(a) $\mathrm{Th}(I)$ is a complete propositional theory.

(b) For any propositional form $A \in \mathrm{VF}(P)$ the following conditions are equivalent:

  (i) $I(A) = 1$

  (ii) $\mathrm{Th}(I) \vdash A$

  (iii) $\mathrm{Th}(I) \models A$

Now, we can prove the original form of the Completeness Theorem. We state it in a way comprising the Soundness Theorem, as well.

Completeness Theorem. *Let $T \subseteq \mathrm{VF}(P)$ be a theory. Then, for every propositional form $B \in \mathrm{VF}(P)$, $T \models B$ if and only if $T \vdash B$.*

Demonstration. If $T \vdash B$ then $T \models B$ by the Soundness Theorem. To show the converse, assume that $T \models B$, nevertheless $T \nvdash B$. By the Theorem on Proof by Contradiction, this means that the theory $T \cup \{\neg B\}$ is consistent. Then, according to the Alternative

Version of the Completeness Theorem, $T \cup \{\neg B\}$ has an interpretation I. Then I is an interpretation of the theory T such that $I(\neg B) = 1$, i.e., $I(B) = 0$. However, since $T \models B$, we have $J(B) = 1$ for every interpretation J of T; in particular, $I(B) = 1$. This contradiction proves that $T \vdash B$.

Finally, let us record the following consequence of the Completeness Theorem.

Compactness Theorem. *Let $T \subseteq \mathrm{VF}(P)$ be a theory. Then T has an interpretation if and only if every finite subtheory T_0 of T has an interpretation.*

Demonstration. By the Completeness Theorem, T has an interpretation if and only if T is consistent. Similarly, every finite subtheory $T_0 \subseteq T$ has an interpretation if and only if every finite subtheory $T_0 \subseteq T$ is consistent. Thus it is enough to realize that T is consistent if and only if every finite subtheory T_0 of T is consistent. Obviously, if T is consistent then so are all its subtheories (and not just the finite ones). The other way round, if T is inconsistent, then any proofs of some couple of contradicting propositional forms B, $\neg B$ in T involve just finitely many specific axioms of T. Putting them together we obtain a finite subtheory $T_0 \subseteq T$ which is already contradictory.

We have formulated and proved the Compactness Theorem in Propositional Calculus mainly with the aim to prepare the way for the Compactness Theorem in Predicate Calculus to come later on. However, the Propositional Calculus version of the Compactness Theorem lacks the importance and the plentitude of consequences of its Predicate Calculus version.

Appendix

Axiomatization of Propositional Calculus Using Four Logical Connectives

For completeness sake we include the axiomatization of Propostional Calculus using all the usual logical connectives $\neg$, $\wedge$, $\vee$ and $\Rightarrow$; the remaining connective $\Leftrightarrow$ is introduced via the logical equivalence

$$A \Leftrightarrow B \equiv (A \Rightarrow B) \wedge (B \Rightarrow A)$$

i.e., the left hand expression serves as the abbreviation for the right hand one. The corresponding list of logical axioms consists of ten axiom schemes. The only inference rule is Modus Ponens, again.

Logical axioms. (10 axiom schemes)
For any propositional forms A, B, C, the following propositional forms are logical axioms:

(LAx 1) $A \Rightarrow (B \Rightarrow A)$
(LAx 2) $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$
(LAx 3) $(A \wedge B) \Rightarrow A$
(LAx 4) $(A \wedge B) \Rightarrow B$
(LAx 5) $A \Rightarrow (B \Rightarrow (A \wedge B))$
(LAx 6) $A \Rightarrow (A \vee B)$
(LAx 7) $B \Rightarrow (A \vee B)$
(LAx 8) $((A \Rightarrow C) \wedge (B \Rightarrow C)) \Rightarrow ((A \vee B) \Rightarrow C))$
(LAx 9) $((A \Rightarrow B) \wedge (A \Rightarrow \neg B)) \Rightarrow \neg A)$
(LAx 10) $A \vee \neg A$

Deduction rule Modus Ponens

(MP) $\dfrac{A,\ A \Rightarrow B}{B}$ (from A and $A \Rightarrow B$ infer B)

Exercise. Show that all the above logical axioms are tautologies and explain their intuitive meaning.

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